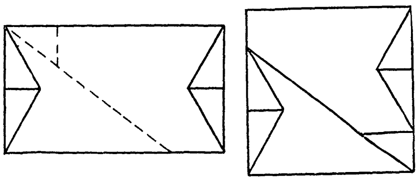
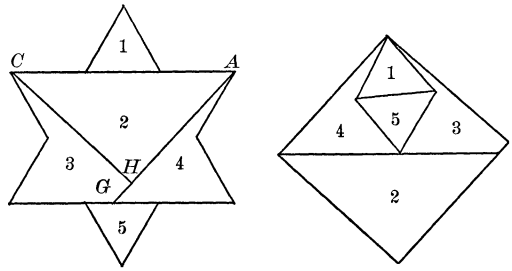
## SOLUTION BY E. B. ESCOTT, Chicago, Illinois.

Find the side of the equivalent square. Draw $AG$ from one point of star equal to the side of the square. Draw $CH$ perpendicular to $AG$. Then cut and arrange the pieces as in the figure.






The proposer gave a similar solution and the accompanying figures to illustrate the newspaper solution and his own.—EDITORS.

**2819 [1920, 134]. Proposed by B. F. FINKEL, Drury College.**

Find the equation of the envelope of the system of circles inscribed in a triangle with a given base and a given vertical angle.

## I. SOLUTION AND REMARKS BY H. S. UHLER, Yale University.

Let the rectangular coördinates of the ends of the base of the triangle be $(b, 0)$ and $(-b, 0)$, let $2\phi$ be the constant vertical angle, and let $\gamma$ denote the angle which the bisector of this angle makes with the positive direction of the $x$-axis.

Since the slope-angles of the bisectors of the base angles at $(b, 0)$ and $(-b, 0)$ are respectively $\frac{1}{2}(\pi + \gamma + \phi)$ and $\frac{1}{2}(\gamma - \phi)$, the equations of these lines may be written

$$y = -(x - b) \cot \tfrac{1}{2}(\gamma + \phi),$$

$$y = (x + b) \tan \tfrac{1}{2}(\gamma - \phi).$$

Solving for $x$ and $y$, the coördinates of the center of the inscribed circles are found to be

$$\left.\begin{aligned} x_c &= b \cos \gamma / \cos \phi, \\ y_c &= b\,(\sin \gamma - \sin \phi)/\cos \phi. \end{aligned}\right\} \qquad (1)$$

Since the radius of the inscribed circle equals $y_c$, the equation of this circle is

$$x^2 + y^2 - 2x_c x - 2y_c y + x_c^2 = 0. \qquad (2)$$

Differentiating equation (2) with respect to $\gamma$, and substituting the values of $dx_c/d\gamma$ and $dy_c/d\gamma$ as obtained from equations (1), we obtain

$$\cos \phi \sin \gamma \cdot x - \cos \phi \cos \gamma \cdot y - b \sin \gamma \cos \gamma = 0. \qquad (3)$$

Solving the equations (2) and (3) for $x$ and $y$, with due regard to equations (1), the parametric equations of the envelope are found to be

$$x = x_c, \qquad y = 0. \qquad (4)$$

$$\left.\begin{aligned} x &= b[2(\sin \gamma - \sin \phi) \sin \gamma + 1] \cos \gamma / \cos \phi, \\ y &= 2b(\sin \gamma - \sin \phi) \sin^2 \gamma / \cos \phi. \end{aligned}\right\} \qquad (5)$$

Equations (4) signify the base of the triangle. This branch of the locus is obviously generated by the lowest point of the inscribed circle as it rolls along the base of the triangle.

The rectangular equation of the other branch may be obtained as follows. Write equation (3) as

$$\cos \gamma = x \cos \phi \sin \gamma / (b \sin \gamma + y \cos \phi)$$

then square the two members and use $\cos^2\gamma = 1 - \sin^2\gamma$ to obtain

$$b^2s^4 + 2bcy\cdot s^3 + [c^2(x^2+y^2) - b^2]s^2 - 2bcy\cdot s - c^2y^2 = 0, \tag{6}$$

where $c \equiv \cos\phi$ and $s \equiv \sin\gamma$.

In the same notation, the second one of equations (5) may be written

$$2bs^3 - 2abs^2 - cy = 0, \tag{7}$$

where $a \equiv \sin\phi$.

Employing Sylvester's dialytic (or any other) method to eliminate $s$ from equations (6) and (7) it will be found that

$$4(\xi^2 + 6av)(3\xi + 4a^2) - (2a\xi - 9v)^2 = 0, \tag{8}$$

where $u \equiv cx/b$, $v \equiv cy/b$, and $\xi \equiv u^2 + v^2 + 2av - 1$.

Since nothing is gained by expanding equation (8) into an explicit function of $x$ and $y$, the solution may be considered as formally complete. Nevertheless it may not be superfluous to call attention to the fact that the act of squaring did not introduce any spurious factors into equation (8), that is, this equation is the simplest non-parametric rational rectangular form of which the upper branch of the locus is susceptible. For, this process amounts to multiplying equation (3), the left member of which is

$$csx - (bs + cy)\cos\gamma,$$

by the rationalizing factor

$$csx + (bs + cy)\cos\gamma,$$

obtained from (3) by changing the sign of $\cos\gamma$, which replaces $\gamma$ by $\pi - \gamma$. Since $s$ is not influenced by supplementary angles, and as the locus in question is symmetrical with respect to the $y$-axis, the introduction of the rationalizing factor should have no other influence than that of repeating or "double-laying" the points on the curve. In conclusion, the very simple but significant relation $dy/dx = \tan 2\gamma$ may be noted.

DISCUSSION: Writing $\omega \equiv 3\sin\gamma - 2a$, equations (5) lead to

$$\frac{dx}{d\gamma} = b\omega\cos 2\gamma/c, \qquad \frac{dy}{d\gamma} = b\omega\sin 2\gamma/c,$$

$$\frac{d^2x}{d\gamma^2} = \frac{b}{c}\left(\cos 2\gamma\cdot\frac{d\omega}{d\gamma} - 2\omega\sin 2\gamma\right), \qquad \frac{d^2y}{d\gamma^2} = \frac{b}{c}\left(\sin 2\gamma\cdot\frac{d\omega}{d\gamma} + 2\omega\cos 2\gamma\right).$$

| $\gamma$ | $x$ | $y$ | $dy/dx$ | $d^2y/dx^2$ | $\rho$ |
|---|---|---|---|---|---|
| $\frac{\pi}{2}$ | $0$ | $2b(1-a)/c$ | $0$ | $\frac{-2c}{b(3-2a)}$ | $\frac{b(3-2a)}{2c}$ |
| $\frac{\pi}{4}$ | $b(\sqrt{2}-a)/c$ | $\frac{b(1-a\sqrt{2})}{c\sqrt{2}}$ | $\infty$ | $\infty$ | $\frac{b(3\sqrt{2}-4a)}{4c}$ |
| $\phi$ | $b$ | $0$ | $\tan 2\phi$ | $\frac{2\cot\phi}{b\cos^3 2\phi}$ | $ab/(2c)$ |
| $\sin^{-1}(\frac{2}{3}a)$ | $bR^3/(27c)$ | $-8a^3b/(27c)$ | $\frac{4aR}{9-8a^2}$ | $\infty$ | $0$ |
| $0$ | $b/c$ | $0$ | $0$ | $-c/(ab)$ | $-ab/c$ |
| $-\frac{\pi}{4}$ | $b(\sqrt{2}+a)/c$ | $\frac{-b(1+a\sqrt{2})}{c\sqrt{2}}$ | $\infty$ | $\infty$ | $\frac{-b(3\sqrt{2}+4a)}{4c}$ |
| $-\frac{\pi}{2}$ | $0$ | $-2b(1+a)/c$ | $0$ | $\frac{2c}{b(3+2a)}$ | $\frac{-b(3+2a)}{2c}$ |

Direct substitution in the formula

$$\frac{d^2y}{dx^2} = \left(\frac{dx}{d\gamma}\cdot\frac{d^2y}{d\gamma^2} - \frac{dy}{d\gamma}\cdot\frac{d^2x}{d\gamma^2}\right)\Big/\left(\frac{dx}{d\gamma}\right)^3$$

gives

$$\frac{d^2y}{dx^2} = 2c/(b\omega \cos^3 2\gamma).$$

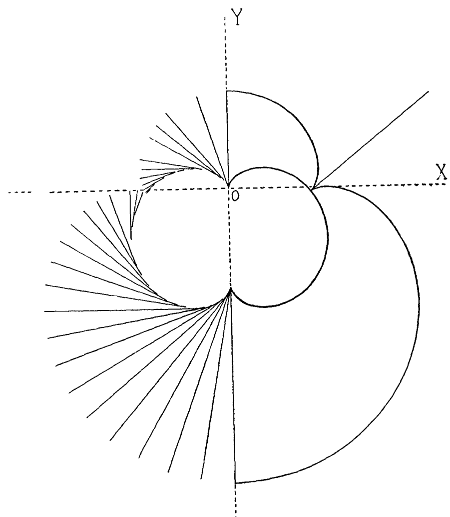
The cartesian formula for the radius of curvature is

$$\rho = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2};$$

hence, for the envelope $\rho = b\omega/(2c)$.

The preceding table shows in condensed form the extreme limits and other characteristic properties of the envelope. $R \equiv (9 - 4a^2)^{1/2}$.

The accompanying diagram was constructed for the case where $2\phi$, the vertical angle of the triangle, equals 60°. Since the loci under consideration are symmetrical with respect to the $y$-axis, the curves have been drawn as continuous lines only on the right side of this axis. The outer curve represents the envelope and the inner one, the evolute of the envelope. The envelope always possesses two cusps the common ordinate of which is negative. A portion of the right cuspidal tangent is shown in the figure. The straight lines in the second and third quadrants represent normals to the envelope. As drawn, each normal extends from a point on the envelope to the corresponding point of tangency on the evolute. Hence, the length of each line gives the radius of curvature of the envelope. These lengths also signify the distances measured along the evolute from the point where it passes through the cusp of the envelope to the point of tangency of the chosen normal.




To show that the envelope always has a cusp at the point whose coördinates are given in the fourth horizontal line of the table, we may proceed as follows. Let $d\sigma$ denote an element of arc of the curve, so that

$$\left(\frac{d\sigma}{d\gamma}\right)^2 = \left(\frac{dx}{d\gamma}\right)^2 + \left(\frac{dy}{d\gamma}\right)^2 = \frac{b^2\omega^2}{c^2}, \qquad \text{or} \qquad \frac{d\sigma}{d\gamma} = \frac{b\omega}{c} = \frac{b}{c}(3 \sin \gamma - 2a).$$

Since this expression vanishes and changes sign for $\gamma = \sin^{-1}(\frac{2}{3}a)$, it is evident that the envelope has a simple cusp at this point.

The terms of the equation

$$d\sigma = \frac{3b}{c} \sin \gamma d\gamma - \frac{2ab}{c} d\gamma$$

can be integrated at once between definite limits and hence the lengths of chosen arcs of the envelope can be determined without difficulty. It will be left to the reader to elaborate this part of the discussion.

Attention will now be turned to the evolute of the envelope. As $dy/dx = \tan 2\gamma$ for the envelope, the equation of the normal at the point $(x', y')$ is

$$y - y' = -(x - x') \cot 2\gamma$$

where $x'$ and $y'$ are given by equations (5). Substituting these values of $x'$ and $y'$, and reducing, we obtain

$$c \cos 2\gamma \cdot x + c \sin 2\gamma \cdot y - b(1 - 2a \sin \gamma) \cos \gamma = 0. \tag{9}$$

Differentiating equation (9) with respect to $\gamma$ we find

$$2c \sin 2\gamma \cdot x - 2c \cos 2\gamma \cdot y - b(\sin \gamma + 2a \cos 2\gamma) = 0. \tag{10}$$

Solving equations (9) and (10) for $x$ and $y$, the parametric equations of the evolute are found to be

$$\left.\begin{aligned} x &= b \cos^3 \gamma/c \\ y &= b(\omega - 2 \sin^3 \gamma)/(2c). \end{aligned}\right\} \tag{11}$$

Eliminating $\gamma$ from equations (11), and introducing the abbreviations defined above, the cartesian formula for the evolute comes out as

$$27u^2 = [4u^2 + 4(v + a)^2 - 1]^3. \tag{12}$$

By direct substitution of the coördinates of the two cusps of the envelope,

$$u = \pm R^3/27, \qquad v = -8a^3/27,$$

it is found that equation (12) is satisfied identically. Hence,—as is true in general,—the evolute passes through the cusps of the original curve.

When $\gamma = \pi/2$ equations (11) give a point on the $y$-axis having the ordinate $b(1 - 2a)/(2c)$. The last expression shows that this point will lie above, or on, or below, the base of the given triangle according as $\phi$ is less than, or equal to, or greater than 30°, respectively. That this point is a simple cusp of the evolute follows at once from the equations

$$\frac{dx}{d\gamma} = -\frac{3b}{2c}\cos\gamma\sin 2\gamma, \qquad \frac{dy}{d\gamma} = \frac{3b}{2c}\cos\gamma\cos 2\gamma, \qquad \frac{d\sigma}{d\gamma} = \frac{3b}{2c}\cos\gamma,$$

since $\cos\gamma$ passes through zero and changes sign when $\gamma = \pi/2$.

It will not be necessary to investigate the nature of the lower point of the evolute on the $y$-axis because the occurrence of $(v + a)^2$ in equation (12) shows that this locus is symmetrical with respect to the horizontal line $v = -a$ or $y = -ab/c$.

## II. Solution[1] by Otto Dunkel, Washington University.

The locus of the center of the inscribed circle is a circle passing through the extremities of the base of the triangle, with center at the lowest point of the circumscribed circle, and with radius equal to $b \sec \phi$.

Let $A'OA$ be the diameter parallel to the base of the triangle. $C$ being the center of the variable circle, angle $AOC$ will be equal to $\gamma$, and the radius of the variable circle will be

$$\frac{b}{\cos\phi}(\sin\gamma - \sin\phi).$$

Applying formula (1) [*1920*, 225],

$$\frac{1}{\delta} + \frac{1}{\delta'} = \frac{2}{R\cos\omega},$$

where $\delta = \infty$, $R = b\sec\phi$, and the angle there called $\omega$ is $90° - \gamma$, we have

$$CF' = \delta' = \frac{b\sin\gamma}{2\cos\phi}.$$

It is therefore easy to construct the point $F'$ of the evolute. Hence the radius of curvature $PF' = \rho = (R\sin\gamma)/2 + R(\sin\gamma - \sin\phi) = R(3\sin\gamma - 2\sin\phi)/2$. Hence the envelope has a cusp where $\sin\gamma = \frac{2}{3}\sin\phi$. There are then two cusps lying upon the evolute, corresponding to the angles $\gamma < \phi$ and $180° - \gamma$. If $C$ is taken at $A$, $P$ will lie on the base produced which forms the tangent at $P$. Hence this line cuts the envelope in two distinct points and in a pair of points of tangency, in all six points. This serves to illustrate how to determine synthetically the important properties of the envelope and to construct it point by point.

Also solved by William Hoover, Joseph Rosenbaum, and F. L. Wilmer.

[1] This is an application of the article on "The Relation of Caustics to Certain Envelopes" printed above. The notation of the previous solution is here employed and the presentation is somewhat condensed.